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The lee-wave régime for a slender body in a rotating flow. Part 2

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The axisymmetric motion of an inviscid, rotating liquid over a prescribed stream surface, say S, is constructed from assumed values of the velocity and azimuthal vorticity on S. The hypothesis of unseparated flow, which implies continuity of the vorticity on S, is shown to imply that: (*a*) the azimuthal vorticity and azimuthal circulation (relative to the basic flow) must be simply proportional to the perturbation stream function in the exterior of S; (*b*) the exterior field exhibits a dipole behaviour far upstream of the body, thereby satisfying Long's hypothesis of no upstream disturbance.

1. Introduction

We consider the axisymmetric flow of an unbounded, inviscid, rotating liquid past a prescribed stream surface, say S, of axial length l and maximum diameter δl. Let U and Ω denote the translational and angular velocities of the basic flow and

$$L \equiv U/(2\Omega) \tag{1.1}$$

the *intrinsic scale*. The flow is characterized by the *inverse Rossby number*,

$$k \equiv 2\Omega l/U = l/L, \tag{1.2}$$

which appears as the length of the body relative to L, and the slenderness ratio, δ. We assume that $k\delta$ has a maximum order of magnitude of unity.

The boundary-value problem posed by the preceding description is considered in some detail in an earlier paper (Miles 1969*a*, hereinafter referred to as I, followed by the appropriate section or equation number) on the basis of Long's (1953) hypothesis that the flow is uniform far upstream of the body. This hypothesis, which precludes the existence of a Taylor column, is controversial (see I for discussion and references), but the prevailing opinion appears to be expressed by Greenspan (1968, p. 224): "Though the last word has yet to be spoken, it seems that a columnar formation is an intrinsic feature of rotating flows and cannot be dismissed. The assumption of no upstream disturbance must be interpreted instead as an approximation appropriate to certain situations." We show, in the following development, that a columnar formation cannot appear in unseparated flow, in which all particles on the stream surface S originate on the upstream axis.

The analysis of I is based on the representation of S by an axial distribution of dipoles of density $f(x)$, which is determined implicitly by an integral equation. Long's hypothesis is supported, in this context, by the solution of the initial-value problem for a dipole in a rotating flow (Miles 1969*b*), which yields a steady-state limit equivalent to that of I. This support is, however, deficient in two respects: (*a*) the solution of the initial-value problem neglects second-order perturbations in the equations of motion; (*b*) the representation of S by an axial distribution of dipoles appears to imply that the exterior solution can be continued analytically into the interior of S (Stewartson, private communication). We avoid these difficulties herein by constructing the steady-state solution in the exterior of S in terms of the tangential velocity and azimuthal vorticity on S. We then show that continuity of this vorticity, which is implied by the hypothesis of unseparated flow, implies that the solution must be of Long's type, in which both the radial moment of the azimuthal vorticity and the azimuthal circulation relative to the basic flow are simply proportional to the perturbation stream function; this, in turn, implies a dipole behaviour far upstream of the body. The resulting problem then is reduced to an integral equation for the meridional velocity on S.

2. Equations of motion

We choose L and U as scales of length and velocity and pose the position, velocity and vorticity vectors in the forms

$$\mathbf{r} = L\{x, r, 0\}, \tag{2.1}$$

$$\mathbf{v} = Ur^{-1}\{\Psi_r, -\Psi_x, \Gamma\}, \tag{2.2}$$

and

$$\boldsymbol{\omega} = 2\Omega r^{-1}\{\Gamma_r, -\Gamma_x, \chi\} \equiv \nabla \times \mathbf{v}, \tag{2.3}$$

where the triad $\{-,-,-\}$ comprises the axial, radial and azimuthal components of a vector, subscripts imply partial differentiation, Ψ is the Stokes stream function, Γ is the azimuthal circulation, and χ/r is the azimuthal vorticity. We also introduce the column matrix ($\boldsymbol{\psi}$ is *not* a vector in the polar co-ordinate space)

$$\boldsymbol{\psi}(x,r) = \{\psi, \gamma, \chi\} \equiv \{\Psi - \tfrac{1}{2}r^2, \Gamma - \tfrac{1}{2}r^2, \chi\}, \tag{2.4}$$

where ψ and γ are perturbations of Ψ and Γ with respect to the uniform flow. We emphasize that our notation differs from that of I both in the choice of characteristic length (L herein *vs.* l in I) and in the sign of ψ. The present choice of sign simplifies the subsequent specialization of the solution; see (3.4) below.

The equations satisfied by Ψ and Γ are developed by Batchelor (1967, §7.5), and may be transformed to

$$\mathscr{D}\psi + \chi = 0, \tag{2.5a}$$

$$\partial(\Gamma, \Psi)/\partial(x,r) = 0, \tag{2.5b}$$

and

$$\partial(\chi, \Psi)/\partial(x,r) + 2r^{-1}(\Psi_x\chi - \Gamma\Gamma_x) = 0, \tag{2.5c}$$

where

$$\mathscr{D} \equiv \partial_x^2 + r\partial_r r^{-1}\partial_r, \tag{2.6}$$

∂_x and ∂_r imply partial differentiation, and the first terms in each of (2.5*b*) and (2.5*c*) are Jacobians. Neglecting terms of second order in ψ, γ, and χ in (2.5*b*, *c*) on the alternative hypotheses that they are either small or vanish identically, we obtain the linear equations

$$\mathscr{D}\psi+\chi = 0, \quad \psi_x-\gamma_x = 0, \quad \gamma_x-\chi_x = 0. \qquad (2.7\,a,b,c)$$

We seek the solution of (2.5) for the flow in the exterior of a closed stream surface S of axial length k with an upstream stagnation point at $x = r = 0$. Invoking this definition of S and the requirements that the velocity, vorticity, and perturbations in the total linear and angular momenta of the fluid be bounded, we obtain

$$\boldsymbol{\psi}(x,0) = 0 \quad (x < 0 \text{ or } x > k), \qquad (2.8a)$$

$$\Psi(x,r) = 0 \quad \text{on } S \quad [0 \leqslant x \leqslant k, r = R(x)] \qquad (2.8b)$$

and

$$\boldsymbol{\psi}(x,r) \to 0 \quad ((x^2+r^2)^{\frac{1}{2}} \to \infty). \qquad (2.8c)$$

3. General solution

The most general solution of (2.5) is governed by (2.5*a*) and the integrals [obtained from (2.5*b*, *c*); see Batchelor (1967, §7.5)]

$$\Gamma = \Gamma(\Psi) \quad \text{and} \quad \chi = \Gamma(\Psi)\Gamma'(\Psi)-r^2H'(\Psi), \qquad (3.1\,a,b)$$

where H is the dimensionless, total head. Invoking (2.8*a*, *b*) and the fact that, by definition, particles on S originate on $r = 0$, we obtain $\Gamma(0) = 0$. Invoking this condition in (3.1*b*) and combining the result with (2.8*b*), we obtain the boundary condition(s)

$$\boldsymbol{\psi} = -\{\tfrac{1}{2}, \tfrac{1}{2}, H_0'\}R^2 \quad \text{on } S \quad [H_0' \equiv H'(\Psi) \text{ on } \Psi = 0]. \qquad (3.2)$$

We remark that H_0' is constant on S by virtue of the hypothesis that the flow is unseparated (see discussion at end of §4).

Choosing

$$\Gamma(\Psi) = \Psi \quad \text{and} \quad H'(\Psi) = \tfrac{1}{2} \qquad (3.3a,b)$$

implies

$$\boldsymbol{\psi} = \{1,1,1\}\psi \equiv \mathbf{I}\psi \qquad (3.4)$$

and

$$\mathscr{D}\psi+\psi = 0, \qquad (3.5)$$

which is a special case of (2.7) for which the second-order terms in (2.5) vanish identically. Conversely, (3.4) implies (3.3) and (3.5). The hypothesis of no upstream disturbance ($\boldsymbol{\psi} \to 0$ as $x \to -\infty$) also implies (3.3) and hence (3.4), but (3.4) does not necessarily imply the absence of an upstream disturbance [in particular, (3.5) admits the cylindrical solution $\psi = rJ_1(r)$].

4. Solution of linear equations

Let V_+ and V_- denote the exterior and interior of S and $\boldsymbol{\psi}_+$ and $\boldsymbol{\psi}_-$ denote exterior and interior solutions of (2.7) such that $\boldsymbol{\psi}_\pm \equiv 0$ in $V_\mp$. We determine $\boldsymbol{\psi}_+$ (subsequently omitting the $+$ subscript) in terms of assumed values of $\boldsymbol{\psi}$ and $\mathbf{v}$ on S and show that it is necessarily of the form (3.4), by virtue of which it satisfies (2.5).

Let $\mathscr{F}$ and $\mathscr{H}$ be the Fourier- and Hankel-transform operators defined by

$$\mathscr{F}(\,) = \int_{-\infty}^{\infty} e^{-i\alpha x}(\,)\,dx, \quad \mathscr{F}^{-1}(\,) = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{i\alpha x}(\,)\,d\alpha, \tag{4.1a,b}$$

and

$$\mathscr{H}(\,) = \int_0^{\infty} J_1(\beta r)(\,)\,dr, \quad \mathscr{H}^{-1}(\,) = r\int_0^{\infty} J_1(\beta r)(\,)\,\beta d\beta. \tag{4.2a,b}$$

Transforming (2.7) on the hypothesis that $\boldsymbol{\psi} \equiv 0$ in V_-, in consequence of which the lower limit of (4.2a) must be replaced by $R(x)$, we obtain

$$\mathbf{M}\mathscr{F}\mathscr{H}\boldsymbol{\psi} = \mathbf{C}, \tag{4.3}$$

where

$$\mathbf{M} = \begin{bmatrix} \alpha^2+\beta^2 & 0 & -1 \\ -i\alpha & i\alpha & 0 \\ 0 & -i\alpha & i\alpha \end{bmatrix} \tag{4.4}$$

is a square matrix, and

$$\mathbf{C} = \mathscr{F}\{\beta J_0\psi - J_1\psi_r + R'J_1\psi_x + (R'J_1\psi)', \quad R'J_1(\psi-\gamma), \quad R'J_1(\gamma-\chi)\}_S \tag{4.5}$$

is a column matrix. All terms in the Fourier operand of $\mathbf{C}$ are evaluated at $r = R(x)$, the prime implies differentiation with respect to x on S [but $\psi_x \equiv \partial_x\psi(x,r)$], and the argument of J_0 and J_1 therein is βR.

Invoking (3.2) and (4.1a) in (4.5), remarking that

$$(r+\psi_r)\,dx - \psi_x dr = r(\mathbf{v}/U).d\mathbf{s} \equiv Rq\,ds \quad \text{on } S, \tag{4.6}$$

where ds is a meridional element of arc and q is the (dimensionless) meridional velocity on S, and integrating the terms in $\beta J_0\psi$, $J_1\psi_r$, and $(R'J_1\psi)'$ by parts, we transform (4.5) to

$$\mathbf{C} = \{(\alpha^2+\beta^2)P - Q, 0, i\alpha(2H_0'-1)P\}, \tag{4.7}$$

where

$$P = \tfrac{1}{2}(i\alpha)^{-1}\int_0^k e^{-i\alpha x} J_1(\beta R)\,R^2R'dx \tag{4.8a}$$

$$= \tfrac{1}{2}\beta^{-1}\int_0^k e^{-i\alpha x} J_2(\beta R)\,R^2dx \tag{4.8b}$$

and

$$Q = \int_S e^{-i\alpha x} J_1(\beta R)\,Rq\,ds. \tag{4.9}$$

Calculating the inverse of $\mathbf{M}$ and taking its product with $\mathbf{C}$, we obtain

$$\mathbf{M}^{-1}\mathbf{C} = \mathbf{I}(2H_0'P - Q)(\alpha^2+\beta^2-1)^{-1} + P\{1, 1, 2H_0'\}. \tag{4.10}$$

Taking the inverse transform of P, as given by (4.8b), we obtain

$$\mathscr{H}^{-1}\mathscr{F}^{-1}P = (4\pi)^{-1}r\int_0^{\infty} J_1(\beta r)\,d\beta\int_{-\infty}^{\infty} d\alpha\int_0^k e^{i\alpha(x-\xi)}J_2[\beta r(\xi)]\,R^2(\xi)\,d\xi \tag{4.11a}$$

$$= \tfrac{1}{2}r\int_0^{\infty} J_1(\beta r)\,J_2[\beta R(x)]\,R^2(x)\,d\beta \tag{4.11b}$$

$$= \tfrac{1}{2}r^2\mathfrak{H}[R(x)-r] \quad (R \equiv 0 \quad \text{if} \quad x < 0 \quad \text{or} \quad x > k), \tag{4.11c}$$

where $\mathfrak{H}$ is Heaviside's step function. It follows that only the first term on the right-hand side of (4.10) contributes to ψ_+, which therefore must be of the form (3.4); this, in turn, implies $H_0' = \frac{1}{2}$ in (3.2). Invoking these conditions, substituting P and Q from (4.8*b*) and (4.9), and then invoking (4.1*b*) and (4.2*b*), we obtain

$$\psi \equiv \mathbf{I}\psi = \mathbf{I}\mathscr{H}^{-1}\mathscr{F}^{-1}(P-Q)(\alpha^2+\beta^2-1)^{-1} \tag{4.12a}$$

$$= -(2\pi)^{-1}\mathbf{I}r\int_0^\infty J_1(\beta r)\,d\beta\int_{-\infty}^{\infty}(\alpha^2+\beta^2-1)^{-1}d\alpha\int_0^k e^{i\alpha(x-\xi)}g(\xi,\beta)\,d\xi, \tag{4.12b}$$

where
$$g(x,\beta) = \beta R J_1(\beta R)\,q(ds/dx)-\tfrac{1}{2}R^2 J_2(\beta R). \tag{4.13}$$

The contributions to the α integral in (4.12*b*) may be derived from the poles at the zeros of $\alpha^2+\beta^2-1$. We infer from a consideration of viscous effects (or in the context of linearized theory, from a consideration of the initial-value problem) that these zeros lie in $\mathscr{I}\alpha > 0$ for $0 < \beta < 1$; in brief,

$$\alpha = \pm(1-\beta^2)^{\frac{1}{2}}+i0+ \quad (0 \leqslant \beta < 1) \tag{4.14a}$$

$$= \pm i(\beta^2-1)^{\frac{1}{2}} \quad (\beta > 1). \tag{4.14b}$$

Closing the contour of integration in $\mathscr{I}\alpha \gtrless 0$ for $x-\xi \gtrless 0$ and invoking Jordan's lemma and Cauchy's residue theorem, we obtain

$$\psi = r\mathfrak{H}(x)\int_0^1 J_1(\beta r)(1-\beta^2)^{-\frac{1}{2}}d\beta\int_0^x g(\xi,\beta)\sin[(1-\beta^2)^{\frac{1}{2}}(x-\xi)]\,d\xi$$
$$-\tfrac{1}{2}r\int_1^\infty J_1(\beta r)(\beta^2-1)^{-\frac{1}{2}}d\beta\int_0^k g(\xi,\beta)\exp[-(\beta^2-1)^{\frac{1}{2}}|x-\xi|]\,d\xi. \tag{4.15}$$

The result (4.15) bears a close resemblence to I (2.10) and points to an error in that result, which, as stated, is not valid for $0 < x < k$ ($0 < x < 1$ in I). Correcting that error (which does not affect any of the other results in I) and allowing for the difference in the sign of ψ and the choice of characteristic length, we obtain ψ in the form (4.15) with $g(x,\beta)$ replaced by

$$g_I(x,\beta) = \beta^2 f(x), \tag{4.16}$$

where $f(x)$ is the dipole density defined by I(2.2). We emphasize that (4.16) does not imply a *direct* equivalence between $f(x)$ and $g(x,\beta)$, as given by (4.13). The only direct equivalence that can be imposed *a priori* is that the two representations of ψ be identical in V_+ and on S.

Letting $x \to -\infty$ in (4.15), we obtain

$$\psi \sim \tfrac{1}{2}F_1 x^{-1} r J_1(r) \quad (x \to -\infty), \tag{4.17}$$

where
$$F_1 = \int_0^k g(\xi,1)\,d\xi \tag{4.18}$$

is the *dipole moment* defined in I (after allowing for the difference of characteristic length).

We infer from (4.17) that Long's hypothesis of no upstream influence is satisfied under the hypothesis of unseparated flow. This hypothesis enters the analysis crucially through the corresponding assumption that H_0' is constant in (3.2). Separated flow implies a discontinuity in $H'(\Psi)$ across the stream surface that emanates from the separation ring. This, in turn, implies that χ is not simply proportional to r^2 on S and leads to a term on the right-hand side of (4.10) that is proportional to $\Delta\chi(i\alpha)^{-1}(\alpha^2+\beta^2-1)^{-1}$, where $\Delta\chi$ is the jump in χ at the separation point. It then is impossible to deal with the pole $\alpha = 0$ without a more detailed consideration of viscous effects, and it is no longer true that ψ has the form (3.4).

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